Estonian Math Competitions 2007/2008

The Gifted and Talented Development Centre

Tartu 2008

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Mathematics Contests in Estonia

The Estonian Mathematical Olympiad is held annually in three rounds – at the school, town/regional and national levels. The best students of each round (except the final) are invited to participate in the next round. Every year, about 110 students altogether reach the final round.

In each round of the Olympiad, separate problem sets are given to the students of each grade. Students of grade 9 to 12 compete in all rounds, students of grade 7 to 8 participate at school and regional levels only. Some towns, regions and schools also organise olympiads for even younger students. The school round usually takes place in December, the regional round in January or February and the final round in March or April in Tartu. The problems for every grade are usually in compliance with the school curriculum of that grade but, in the final round, also problems requiring additional knowledge may be given.

The first problem solving contest in Estonia took place already in 1950. The next one, which was held in 1954, is considered as the first Estonian Mathematical Olympiad.

Apart from the Olympiad, open contests are held twice a year, usually in October and in December. In these contests, anybody who has never been enrolled in a university or other higher education institution is allowed to participate. The contestants compete in two separate categories: the Juniors and the Seniors. In the first category, students up to the 10th grade are allowed to participate; the other category has no restriction. Being successful in the open contests generally assumes knowledge outside the school curriculum.

According to the results of all competitions during the year, about 20 IMO team candidates are selected. IMO team selection contest for them is held in April or May. This contest lasts two days; each day, the contestants have 4.5 hours to solve 3 problems, similar to the IMO. All participants are given the same problems. Some problems in our selection contest are at the level of difficulty of the IMO but somewhat easier problems are usually also included.

The problems of previous competitions can be downloaded from
`http://www.math.olympiaadid.ut.ee/eng`.

Besides the above-mentioned contests and the quiz "Kangaroo" some other regional competitions and matches between schools are held as well.

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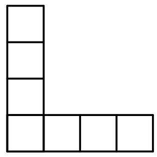
This booklet contains problems that occurred in the open contests, the final round of national olympiad and the team selection contest. For the open contests and the final round, selection has been made to include only original and interesting problems. The team selection contest, containing only original problems, is presented entirely.

Selected Problems from Open Contests

OC-1. An n-boomerang consists of $2n - 1$ unit squares arranged in an L-shape with both legs of length n ($n = 4$ in the figure). Find all integers $n \geqslant 2$ for which there exists a rectangle with integer side lengths that can be partitioned into n-boomerangs. *(Juniors.)*



Answer: the only suitable integer is 2.

Solution. If $n = 2$, the rectangle of size 2×3 can be partitioned into two 2-boomerangs (Fig. 1).

Let us prove that if $n \geqslant 3$ then there are no rectangles that can be partitioned into n-boomerangs. Let (x, y) denote the unit square located in row x and column y, where x and y are positive integers. Denote by $(a, b) - (c, d) - (e, f)$ the boomerang with end-squares in unit squares (a, b) and (e, f) and corner-square in unit square (c, d). Clearly $(1, 1)$ has to be covered by an end-square or a corner-square of some boomerang.

If a boomerang covers $(1, 1)$ with its corner-square then $(2, 2)$ can be covered by another boomerang again with a corner-square or an end-square.

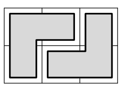


Figure 1

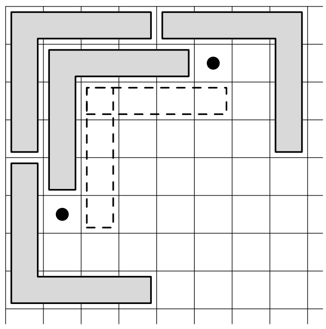


Figure 2

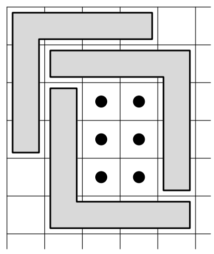


Figure 3

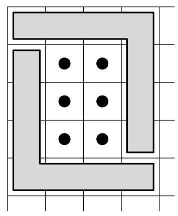


Figure 4

1. If another boomerang covers $(2, 2)$ with a corner-square (Fig. 2) then the boomerangs covering $(n + 1, 1)$ and $(1, n + 1)$ leave the squares $(n + 2, 2)$ and $(2, n + 2)$ empty. For any position of the boomerang covering $(3, 3)$, one of $(n + 2, 2)$ and $(2, n + 2)$ can not be covered anymore.
2. If another boomerang covers $(2, 2)$ with its end-square (Fig. 3) then let it be without loss of generality positioned as $(2, 2) - (2, n + 1) - (n + 1, n + 1)$. The boomerang covering $(3, 2)$ is then positioned as $(3, 2) - (n + 2, 2) - (n + 2, n + 1)$. The second and third boomerang now have an isolated empty rectangle with side lengths less than n between them.

If the first boomerang covers $(1, 1)$ with its end-square (Fig. 4) then we can assume without loss of generality that it is positioned as $(1, 1) - (1, n) - (n, n)$. The boomerang covering $(2, 1)$ must now be positioned as $(2, 1) - (n + 1, 1) - (n + 1, n)$. Those two have an empty rectangle in between that is too small to fit any boomerangs.

OC-2. Do there exist four different integers a, b, c, d, all greater than one, satisfying $\gcd(a, b) = \gcd(c, d)$ and a) $ab = cd$; b) $ac = bd$? *(Juniors.)*

Answer: a) yes; b) no.
Solution 1. a) Let x, y, z and w be arbitrary different pairwise co-prime positive integers. Let $a = xy$, $b = zw$, $c = xz$ and $d = yw$. All these numbers are greater than 1. Then $\gcd(a, b) = \gcd(xy, zw) = 1$ and also $\gcd(c, d) = 1$, whereas $ab = cd = xyzw$.
b) Assume for contradiction that such a, b, c, d exist. Let $s = \gcd(a, b) = \gcd(c, d)$. Write $a = a's$, $b = b's$, $c = c's$, $d = d's$, then $\gcd(a', b') = 1$ and $\gcd(c', d') = 1$. The equation $ac = bd$ becomes $a's \cdot c's = b's \cdot d's$, equivalently $a'c' = b'd'$. Thus d' divides $a'c'$ and hence, since c' and d' are co-prime, d' divides a'. Analogously, since a' divides $b'd'$ and b' and a' are co-prime, a' divides d'. It follows that $a' = d'$ and $a = d$. This contradicts the assumption that all of a, b, c, d are different.
Solution 2. b) Assume for contradiction that such a, b, c, d exist. Write the equation as

$$\frac{a}{b} = \frac{d}{c}.$$

When we put equal fractions into lowest terms, we get equal fractions (in lowest terms) with equal numerators and also equal denominators. The number we divide by is the greatest common divisor of the numerator and the denominator. Since we are given $\gcd(a, b) = \gcd(c, d)$ the denominators and numerators will be divided through by the same number, that is, the numerators and denominators must be equal to begin with. Thus $a = d$ and $b = c$, contradicting the assumption that a, b, c, d are different.

OC-3. How many 5-digit natural numbers are there such that after deleting any one digit, the remaining 4-digit number is divisible by 7? *(Juniors.)*

Answer: 8.
Solution. Let $M = \overline{abcde}$ be a number with the required property. By deleting a and b we get $A = \overline{bcde}$ and $B = \overline{acde}$, respectively. Since they are divisible by 7, so is their difference $B - A = 1000(a - b)$, hence $a - b$ is divisible by 7, hence a and b are congruent modulo 7. Analogously, we have that b and c, that c and d, and finally that d and e are congruent modulo 7. Thus all the digits are congruent modulo 7.
If M has digits that are at least 7, we can subtract 7 from each such digit to obtain a new number M'. It is easy to see that M satisfies the condition in the problem if and only if M' does. Since all digits give the same remainder, we are left to consider $\overline{xxxxx}$ where $0 \leqslant x \leqslant 6$. By deleting a digit we get $\overline{xxxx} = x \cdot 1111$ that is divisible by 7 only if $x = 0$. Indeed, 1111 and 7 are co-prime. Thus every digit of M is either 0 or 7. The first two digits must be 7 (since the number has 5 digits and any number we get by deleting a digit has 4 digits), the last three digits can be any of 0 or 7 independently. Thus there are $2 \cdot 2 \cdot 2 = 8$ suitable numbers.

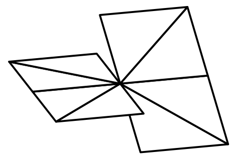
OC-4. A magician wants to do the following trick, using an n-year-old volunteer from the audience. On a board, the magician writes n different positive integers in a row. Now, between every two consecutive integers, the volunteer writes the difference of the inverses of the left-hand and right-hand numbers. He finds that all the differences are equal. Show that the magician can do the trick with every volunteer who is at least 2 years old. *(Juniors.)*

Solution. If the volunteer is n years old then the magician can pick a number N that is divisible by all the integers from 1 to n, e.g. their least common denominator or product, and write on the board the numbers

$$\frac{N}{1}, \frac{N}{2}, \ldots, \frac{N}{n}.$$

These are different positive integers whose inverses are $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{n}{N}$ respectively. We see that the differences of consecutive numbers are all equal to $-\frac{1}{N}$.

OC-5. A *squaric* is a square that has been divided into 8 equal triangles by perpendicular bisectors of its sides and its diagonals. Each of those lines divides the squaric into two parts; we can take one of the parts and reflect it over a second dividing line that is perpendicular to the original line (equivalently, we rotate along the line by 180° in space). Every triangle has been coloured by one of four colours and there are two triangles of each colour. Show that regardless of the initial colouring, the squaric can be taken to an end position where at every side of the square both triangles have the same colour. *(Juniors.)*



Solution. Denote the positions of triangles by numbers 1 to 8 and axes of reflection by letters x, u, y, v as seen in Fig. 5. Additionally, let A, B, C, D be the colours used.

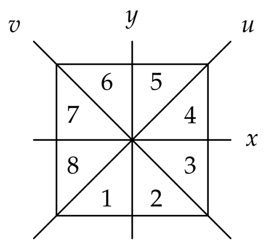


Figure 5

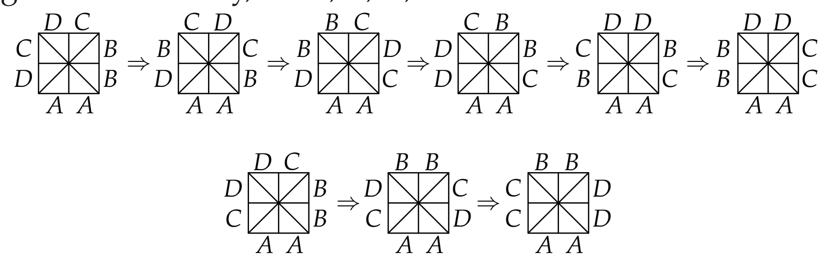


Figure 6

Assume without loss of generality that the triangle in position 1 has colour A. If the other triangle of colour A is not in position 2, we take it there by reflections that leave the triangle in position 1 fixed.

Position of colour A triangle	Axes of reflection
3	u, y, x
4	u, x
5	x
6	y, x
7	v, y, x
8	v, x

Now the triangles at the bottom side of the squaric have the same colour.
Assume without loss of generality that the triangle at position 3 has colour B. If the other triangle of colour B is not located at position 4, we take it there as follows, always leaving the triangle in position 1 unmoved:

Position of colour B triangle	Axes of reflection
5	y, v, y
6	v, y
7	y
8	v, y, v, y

Now the bottom and right sides of the squaric have triangles of suitable colours.
Finally assume without loss of generality that the triangle in position 5 has colour C. If the other triangle coloured C lies in position 6, we are done, since then the triangles at positions 7 and 8 must have colour D. If the other triangle of colour C does not lie in position 6, we use the following reflections to solve the squaric, always leaving the triangle in position 1 unmoved (cf. Fig. 6):

Position of colour C triangle	Axes of reflection
7	y, u, y, v, y
8	u, y

OC-6. Is it true that every polynomial $P(x) = a_m x^m + \ldots + a_1 x + a_0$ with integer coefficients whose value $P(z)$ for every integer z is a composite number can be written as $P(x) = Q(x) \cdot R(x)$, where Q and R are polynomials with integer coefficients, neither of which is constantly 1 or -1? *(Seniors.)*

Answer: no.
Solution. Let $P(x) = x^2 + x + 4$. If a is any integer then $P(a) = a^2 + a + 4 = a(a+1) + 4$. One of a and $a + 1$ has to be even, thus $P(a)$ is even. Since a and $a + 1$ cannot have opposite signs, $a(a + 1)$ is non-negative, thus $P(a) \geqslant 4$. Thus $P(a)$ is composite.
Write $P(x) = Q(x) \cdot R(x)$, where Q and R are polynomials with integer coefficients. Since the leading coefficient of P is 1, the leading coefficients of Q and R are both 1 or both -1. So if Q or R is constant it has to be 1 or -1. If neither is a constant, since P is a square polynomial, Q and R have to be linear polynomials. But P has no roots, hence it is not a product of linear polynomials. Therefore $P(x) = x^2 + x + 4$ cannot be written as a product of polynomials both different from -1 and 1.

OC-7. Let O be the circumcentre of triangle ABC. Lines AO and BC intersect at point D. Let S be a point on line BO such that $DS \parallel AB$ and lines AS and BC intersect at point T. Prove that if O, D, S and T lie on the same circle, then ABC is an isosceles triangle. *(Seniors.)*

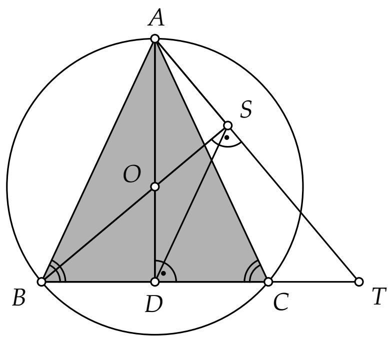


Figure 7

Solution. We have that OAB is an isosceles triangle; so is OSD since DS and AB are parallel (Fig. 7). It follows that the triangles OAS and OBD are equal, using that $|OA| = |OB|$, $|OS| = |OD|$ and $\angle SOA = \angle DOB$. Thus $\angle OSA = \angle ODB$, from which it follows that $\angle OST = \angle ODT$. The points O, D, S, T are located on a circle, so that the points D and S that are symmetric with respect to the perpendicular bisector of the segment AB are located on different sides of the line OT. It follows that OST and ODT are opposite angles of the inscribed quadrilateral $ODTS$ and their sum is $180°$. Thus $\angle OST = \angle ODT = 90°$. The altitude from vertex A of the triangle ABC goes through the circumcentre of ABC, so it is also the perpendicular bisector of BC. It is possible only if $|AB| = |AC|$.

Remark. If ABC is a right triangle, the validity of the claim depends on definitions used. By the usual high-school definition a right triangle does not satisfy the assumptions of the problem: if the right angle is at A, then $S = D$ and we cannot talk about line DS; if the right angle is at B there is no intersection point T; if the right angle is at C, the lines DS and AB would coincide but coinciding lines are not considered parallel at school.

OC-8. Wolf and Fox play the following game on a board with a finite number of unit squares. In the beginning, all squares are white and empty. First, Wolf picks up a game piece from a pile, and either places it on a white square, paints this square gray and removes all the other pieces from the board, or places it on an empty gray unit square. Then, Fox makes a move by the same rules, only her colour is red and not gray. The players continue taking turns and the last player to make a move wins (assume there is an infinite supply of game pieces). Who wins if both play optimally? *(Seniors.)*

Answer: Wolf.

Solution 1. Let Wolf have the following strategy. If there are white squares on the board he will place a piece on one (and colour it gray), otherwise on an empty gray square.

Let us prove this is a winning strategy. Since Wolf starts occupying white squares and colours them at every move, then after each of his moves there are more gray than red squares. After every move of Fox there are at least as many gray squares than red. This is true until there are no more white squares. The square coloured last contains a piece, all the other squares are empty and the game continues so that each player places pieces on empty squares.

If the last white square is coloured by Wolf there will be more gray squares than red. Although the square coloured last contains a piece and he cannot move there anymore, there are at least as many gray squares left as red. This means Wolf can move after every move of Fox. If the last square is coloured by Fox, it will contain a piece and she cannot move there afterwards. Therefore Wolf will have more empty gray squares than Fox has empty red ones. In both cases Wolf gets to make the last move.

Solution 2. Wolf can, after the first move of each player, colour a white square on his second move, and after that copy moves of Fox.

Solution 3. It is clear that the game always ends, so somebody has a winning strategy.

Suppose Fox has a winning strategy. At the first move no player has a choice (they colour a white square). At the second move Wolf has two choices.

- He colours a new white square. Then there are two gray squares, one containing a piece, and one empty red square on the board. By assumption, Fox has a winning strategy.
- He places a piece on the gray square. Fox can only colour a new white square. After Fox moves there are two red squares, one containing a piece and one empty gray square on the board. By symmetry now Wolf has a winning strategy.

Contradiction.

OC-9. The teacher gives every student a triple of positive integers. First, every student has to reduce the second and third number by dividing them by their greatest common divisor, then reduce the first and third number of the resulting triple by dividing them by their greatest common divisor, and finally, reduce the first and second number of the new triple by dividing them by their greatest common divisor. Then, everybody has to multiply the numbers in the final triple and tell the result to the teacher. It is known that the initial triples only differ by the order of numbers. Find the greatest possible number of different correct answers that the students could get. *(Seniors.)*

Answer: 3.

Solution. Let (a, b, c) be the initial triple and

$$d_1 = \gcd(b, c), \qquad d_2 = \gcd\left(a, \frac{c}{d_1}\right), \qquad d_3 = \gcd\left(a, \frac{b}{d_1}\right).$$

After the first and second division, we get triples $\left(a, \frac{b}{d_1}, \frac{c}{d_1}\right)$ and $\left(\frac{a}{d_2}, \frac{b}{d_1}, \frac{c}{d_1 d_2}\right)$, respectively.

Let us prove that $\gcd\left(\frac{a}{d_2}, \frac{b}{d_1}\right) = d_3$. Since $\frac{b}{d_1}$ and $\frac{c}{d_1}$ are co-prime, their divisors d_2 and d_3 are co-prime. Since d_3 divides a, it hence divides $\frac{a}{d_2}$. Since d_3 divides $\frac{b}{d_1}$, it also divides $\gcd\left(\frac{a}{d_2}, \frac{b}{d_1}\right)$. On the other hand, since $\frac{a}{d_2}$ divides a, clearly $\gcd\left(\frac{a}{d_2}, \frac{b}{d_1}\right)$ divides d_3 which proves the claim. Therefore the triple after the third division is $\left(\frac{a}{d_2 d_3}, \frac{b}{d_1 d_3}, \frac{c}{d_1 d_2}\right)$ and the correct answer is $\frac{abc}{d_1^2 d_2^2 d_3^2}$.

If we swap b and c in the initial triple, d_1 is left unchanged and d_2 and d_3 are swapped which leaves the final answer unchanged. Therefore the answer depends only on what we choose as the first component of the triple. Thus there can not be more than 3 different answers.

We get three different correct answers if we pick a triple (p^2qr, pq^2r, pqr^2) where p, q and r are pairwise different primes. Indeed this triple changes as

$$(p^2qr, pq^2r, pqr^2) \mapsto (p^2qr, q, r) \mapsto (p^2q, q, 1) \mapsto (p^2, 1, 1),$$

giving the answer p^2. By changing cyclically the order of the components in the triple, answers q^2 and r^2 are obtained.

OC-10. In a square grid of dimension $m \times n$ where $m, n \geqslant 5$, every square has been coloured black or white. At each step, we can pick some horizontal or vertical strip of width 1 and odd length that contains squares of both colours, and colour all squares in this strip by the colour occurring less in the strip. Show that by these steps we can give all squares the same colour. *(Seniors.)*

Solution 1. Let the grid have m rows and n columns. At first we shall show that the squares in each row can be given the same colour. If n is odd we cover a row by one strip and colour all the squares by the colour that occurs less in that row. If n is even we cover all squares in the row except for the first one by a strip (of odd length) and give them one colour. If the first square is coloured differently from the rest of the row we cover the squares 1, 2 and 3 by a strip and give them the same colour, then we cover the squares 1, 2, 3, 4 and 5 by a strip and colour them by the colour of squares 4 and 5. Now all the squares in that row have the same colour.

Let us now do the same construction for columns. After that the squares in each column have the same colour, but also all the colours are equal since after the first stage all columns looked identical.

Solution 2. At first, prove that a rectangle with dimensions 1×5 that contains squares of both colours can be coloured by each colour. Now we can make the first row monochromatic by sequentially choosing the 1×5 blocks (or not choosing if a block is already monochromatic). Then we can similarly give every column the colour of its first square.

Remark. One can find other solutions, precisely by dividing the grid into at most four sub-grids, with each dimension odd and at least 3, and making the sub-grids and eventually the whole grid monochromatic.

OC-11. We are given different positive integers $a_1, a_2, \ldots, a_n$ where $n \geqslant 3$ and every integer except the first and last one is the harmonic mean of its neighbours. Show that none of the given integers is less than $n - 1$. *(Seniors.)*

Solution. As given, the numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ form an arithmetic progression. By symmetry we may assume that $\frac{1}{a_1} > \frac{1}{a_2} > \ldots > \frac{1}{a_n}$, equivalently $a_1 < a_2 < \ldots < a_n$. Since $a_1, a_2, \ldots, a_n$ are positive integers,

$$\frac{1}{a_2} > \frac{1}{a_2} - \frac{1}{a_n} = (n-2)\left(\frac{1}{a_1} - \frac{1}{a_2}\right) = (n-2) \cdot \frac{a_2 - a_1}{a_1 a_2} \geqslant \frac{n-2}{a_1 a_2}.$$

Hence $\frac{1}{a_2} > \frac{n-2}{a_1 a_2}$, and multiplying both sides by $a_1 a_2$ gives $a_1 > n - 2$. Since a_1 is an integer, $a_1 \geqslant n - 1$. The numbers $a_2, \ldots, a_n$ are greater than a_1 and thus greater than $n - 1$.

OC-12. Two circles are drawn inside a parallelogram $ABCD$ so that one circle is tangent to sides AB and AD and the other is tangent to sides CB and CD. The circles touch each other externally at point K. Prove that K lies on the diagonal AC. *(Seniors.)*

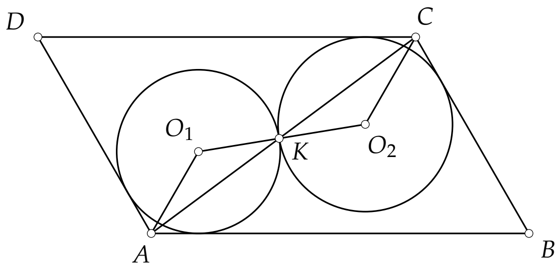


Figure 8

Solution 1. Let O_1 and O_2 be the centres of the first and the second circle, respectively (Fig. 8). Consider the triangles O_1AK and O_2CK. Their angles AO_1K and CO_2K are equal since their sides O_1K and O_2K lie on the same line and the sides O_1A and O_2C are parallel since $\angle O_1AB = \frac{1}{2}\angle DAB = \frac{1}{2}\angle BCD = \angle O_2CD$. As $\angle DAB = \angle BCD$, we have $\frac{|O_1A|}{|O_1K|} = \frac{|O_2C|}{|O_2K|}$. Therefore the triangles O_1AK and O_2CK are similar. Thus $\angle O_1KA = \angle O_2KC$, from which it follows that the points A, K and C are collinear.

Solution 2. Consider the homothety with centre K that takes one of the circles onto the other one. Then the line AB is taken to the line CD and the line AD is taken to the line CB. Thus the intersection point A of the lines AB and AD goes to the intersection point C of the lines CD and CB. It follows that the points A, K and C are collinear.

OC-13. Let x and y be arbitrary real numbers.

a) If $x+y$ and $x+y^2$ are rational numbers, can we deduce that x and y are rational numbers?
b) If $x+y$, $x+y^2$ and $x+y^3$ are rational numbers, can we deduce that x and y are rational numbers?

(Seniors.)

Answer: a) no; b) yes.

Solution. a) Pick $x = \dfrac{1-\sqrt{2}}{2}$ and $y = \dfrac{1+\sqrt{2}}{2}$. Then $x+y=1$ and $x+y^2 = \dfrac{5}{4}$. We see that $x+y$ and $x+y^2$ are rational but x and y are not.
b) Let $x+y$, $x+y^2$ and $x+y^3$ be rational. If $y=0$ or $y=1$ then both x and y are rational. If $y \neq 0$ and $y \neq 1$ then the following is also a rational:

$$\frac{(x+y^3)-(x+y^2)}{(x+y^2)-(x+y)} = \frac{y^3-y^2}{y^2-y} = \frac{y(y^2-y)}{y^2-y} = y.$$

Thus $(x+y)-y = x$ is also rational.

OC-14. A sequence (a_n) of natural numbers is given by the following rule:

$$a_n = \frac{\operatorname{lcm}(a_{n-1}, a_{n-2})}{\gcd(a_{n-1}, a_{n-2})} \quad \text{for all } n \geqslant 2.$$

It is known that $a_{560} = 560$ and $a_{1600} = 1600$. Find all possible values of a_{2007}. *(Seniors.)*

Answer: $a_{2007} = 140$ is the only possible value.
Solution. Explore the behaviour of the sequence in general. Note that it is sufficient to consider the behaviour of the sequence for each prime factor separately. We have

$$\frac{\operatorname{lcm}(p^a, p^b)}{\gcd(p^a, p^b)} = \frac{p^{\max(a,b)}}{p^{\min(a,b)}} = p^{|a-b|}.$$

Therefore for each prime factor one may consider the behaviour of the sequence of exponents. Thus explore the properties of the sequence (b_n) defined by $b_n = |b_{n-1} - b_{n-2}|$ for all $n \geqslant 2$. It is easy to see that either all terms of the sequence are even or there is a cycle (even, odd, odd). Thus b_{n+3} and b_n have the same parity. Also observe that $b_{n+3} \leqslant b_n$ for all n.
Consider now the prime factors appearing in given terms. For prime factor 7 one has $b_{560} = 1$ and $b_{1600} = 0$. Since $3 \mid 1601 - 560$, previous observations imply that $b_{1601} = 1$. Then $b_{1602} = |1-0| = 1$. Now the fact that $3 \mid 2007 - 1602$ leads to $b_{2007} = 1$. Hence the exponent of 7 in a_{2007} is equal to 1.
For the prime factor 5, we have $b_{560} = 1$ and $b_{1600} = 2$. Analogously with previous case we obtain $b_{1601} = 1$ and also $b_{1602} = 1$. Hence as before $b_{2007} = 1$, thus the exponent of 5 in a_{2007} is 1.

The prime factor 2 remains. For this $b_{560} = 4$ and $b_{1600} = 6$. Examine the possible values of b_{1601}. Since it must have the same parity as b_{560} and may not be greater than it, the only candidates are 0, 2 and 4. Suppose $b_{1601} = 0$. Then both b_{1601} and b_{1600} are divisible by 6. Taking into account the definition of the sequence (b_n) implies that 6 divides also all previous terms, including b_{560}. This leads to contradiction, thus b_{1601} is not 0. Suppose that $b_{1601} = 2$. Since consequent terms are even, all terms of the sequence must be even. Dividing all terms by 2 leads to sequence, that still satisfies the definition, thus all previously considered observations must be valid. The term b_{560} transforms to 2 and the term b_{1601} to 1, that means they have different parity. This is a contradiction analogously to the cases of previous primes.
The last remaining possibility is $b_{1601} = 4$ (it is easy to see that a corresponding sequence exists). Now performing calculations we obtain $b_{1602} = 6 - 4 = 2$, $b_{1603} = 4 - 2 = 2$, $b_{1604} = 2 - 2 = 0$, $b_{1605} = 2 - 0 = 2$ and further the cycle $(2, 0, 2)$ repeats, therewith the value of terms with the number divisible by 3 is 2. Thus $b_{2007} = 2$, hence the exponent of 2 in a_{2007} is 2.
Since the terms a_{560} and a_{1600} have no other prime factors, taking preceding into account implies that the term a_{2007} neither has other prime factors. Hence the only solution is $a_{2007} = 7 \cdot 5 \cdot 2^2 = 140$.

Selected Problems from the Final Round of National Olympiad

FR-1. On a railway connecting cities A and B, trains run at full speed except for two railway segments, where poor track conditions force them to slow down. If any one of those two segments were repaired, the average speed of a train between A and B would increase by a third. How much would the average speed between A and B increase if both segments were repaired? *(Grade 9.)*

Answer: 2 times.
Solution. Let the train journey between A and B take time t when neither segment is repaired. If the first segment was repaired, the average speed would increase by a third, in other words, $\frac{4}{3}$ times, so the journey would take time $\frac{3}{4}t$. Thus, repairing the first segment would save $\frac{1}{4}t$ time. Similarly, repairing the second segment would save $\frac{1}{4}t$. Repairing both segments would save $\frac{1}{2}t$ and the average speed would increase 2 times.

FR-2. Find all possible values of $\overline{abc} \cdot (a + b + c)$, given that $\overline{bca} = (a + b + c)^3$ and $b \neq 0$. *(Grade 9.)*

Answer: 2008.
Solution. There exist five three-digit cubes: $125 = 5^3$, $216 = 6^3$, $343 = 7^3$, $512 = 8^3$ and $729 = 9^3$. Of these, only 512 satisfies $\overline{bca} = (a + b + c)^3$. Thus, $a = 2$, $b = 5$, $c = 1$ and $\overline{abc} \cdot (a + b + c) = 251 \cdot (2 + 5 + 1) = 2008$.

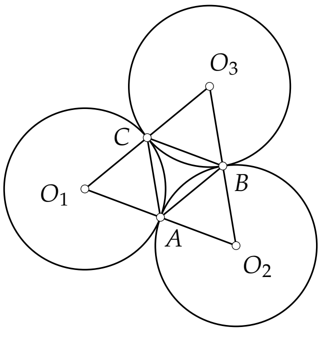


Figure 9

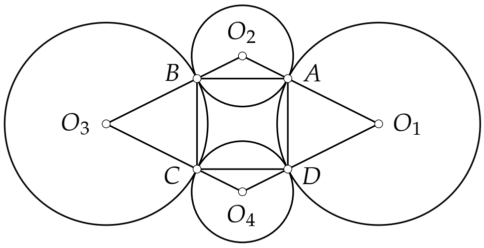
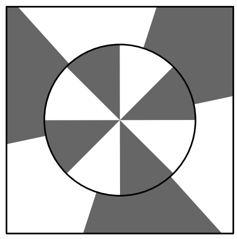


Figure 10

FR-3. a) Circles c_1 and c_2 touch externally at point A, circles c_2 and c_3 touch externally at point B, and circles c_3 and c_1 touch externally at point C. Suppose that triangle ABC is equilateral. Are the radii of c_1, c_2 and c_3 necessarily equal?
b) Circles c_1 and c_2 touch externally at point A, circles c_2 and c_3 touch externally at point B, circles c_3 and c_4 touch externally at point C, and circles c_4 and c_1 touch externally at point D. Suppose that $ABCD$ is a square. Are the radii of c_1, c_2, c_3 and c_4 necessarily equal? *(Grade 9.)*

Answer: a) yes; b) no.
Solution. a) Let O_1, O_2 and O_3 be the midpoints of c_1, c_2 and c_3, respectively (Fig. 9). Triangles O_1CA, O_2AB and O_3BC are isosceles, as each triangle has two radii of the same circle as its two sides. Let $\angle O_1CA = \angle O_1AC = \alpha$, $\angle O_2AB = \angle O_2BA = \beta$ and $\angle O_3BC = \angle O_3CB = \gamma$. Suppose that triangle ABC is equilateral, so $\angle ABC = \angle BCA = \angle CAB = 60^\circ$. As $\angle O_1AC + \angle CAB + \angle O_2AB = 180^\circ$, we have $\alpha + \beta = 120^\circ$. Similarly, $\beta + \gamma = 120^\circ$ and $\gamma + \alpha = 120^\circ$. The last three equations together give $\alpha = \beta = \gamma = 60^\circ$. Thus, triangles O_1CA, O_2AB, O_3BC are equilateral and as ABC is also equilateral, they are in fact equal.
b) Choose the midpoints of the three circles as $O_1(6;0)$, $O_2(0;3)$, $O_3(-6;0)$, $O_4(0;-3)$ (Fig. 10). Then $O_1O_2O_3O_4$ is a rhombus and points $A(2;2)$, $B(-2;2)$, $C(-2;-2)$, $D(2;-2)$ on the sides of the rhombus form a square. Take each vertex of the rhombus to be the midpoint of a circle drawn through the two closest vertices of the square. Then these four circles touch externally at A, B, C, D, yet they do not all have equal radii (e.g., $|O_1A| \neq |O_2A|$).

FR-4. Let n be a positive integer. Rays originating from the midpoint X of a revolving stage divide the stage into $2n+2$ equal sectors, coloured alternatingly black and white ($n = 3$ in the figure). Similarly, equally spaced rays originating from X divide the fixed floor area outside the revolving stage into $2n$ alternatingly black-and-white sectors. Prove that regardless of the position of the revolving stage, there exists a sector of the stage that is completely embraced by a single fixed floor sector of the same colour. *(Grade 9.)*



Solution. Consider the rays dividing the revolving stage into $2n+2$ sectors. Since the rest of the floor is divided into $2n$ sectors, there exist two neighbouring rays that pass through the same floor sector. If the revolving stage sector between those two rays has the same colour as the floor sector, we are done. If, on the other hand, the two sectors are of different colour, then the stage sector symmetrically opposite to the original sector satisfies our conditions. This stage sector is completely embraced by the floor sector symmetrically opposite the original floor sector, however, when turning 180°, the stage sectors change colour $n+1$ times, while the floor sectors change colour only n times, so the two sectors symmetrically opposite to the original sectors are of the same colour.

FR-5. Circles c_1 and c_2 with midpoints O_1 and O_2 intersect at point P. Circle c_2 intersects O_1O_2 at point A. Prove that there exists a circle touching c_1 at P and O_1O_2 at A iff $\angle O_1PO_2 = 90^\circ$. *(Grade 10.)*

Solution. Assume first there exists a circle c touching c_1 at P and O_1O_2 at A (Fig. 11). Let O be the midpoint of c, then line O_1O passes through P. Consider triangles OPO_2 and OAO_2. Clearly $|OP| = |OA|$ and $|O_2P| = |O_2A|$ as the radii of circles c and c_2. Also, the triangles share a third side OO_2, so they are equal. As $\angle OAO_1 = 90^\circ$, we must also have $\angle OPO_2 = 90^\circ$.
Assume now $\angle O_1PO_2 = 90^\circ$. Then line O_2P is perpendicular to radius O_1P and thus touches c_1 at P. As $|O_2P| = |O_2A|$, the line drawn through P perpendicular to O_2P and the line drawn through A perpendicular to O_2A intersect at a point O such that $|OP| = |OA|$. A circle with midpoint O and radius OP then touches c_1 at P and O_1O_2 at A.

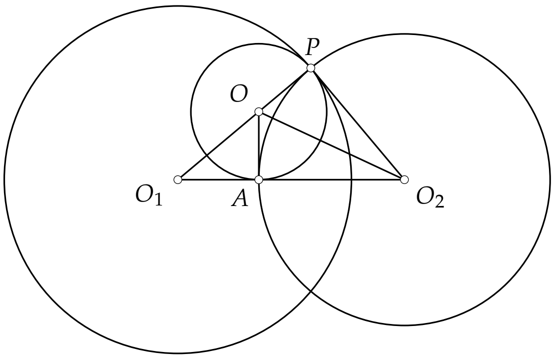


Figure 11

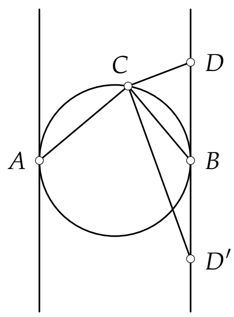


Figure 12

FR-6. Do there exist 5 different points in the plane such that all triangles with vertices at these points are right triangles and

a) no four of the chosen points lie on the same line;
b) no three of the chosen points lie on the same line?

(Grade 10.)

Answer: a) yes; b) no.
Solution 1. a) Choose four vertices of a square and the intersection point of its diagonals.
b) Consider a set of points in the plane such that all triangles with vertices in those points are right triangles and no three points lie on the same line. Choose some two points A and B; all the remaining points then lie either on the circle with diameter AB, or on either line perpendicular with AB drawn through endpoint A or B (Fig. 12). At most four points (including A and B) can lie on the circle, since any two of such three

points must be the two endpoints of some diameter. Also, in addition to A and B, there can be at most one point on either perpendicular.

Suppose now that C and D are two points satisfying our conditions such that C lies on the circle and D lies on one of the two lines, say, on the perpendicular drawn through B. If C and D lie on the same side of AB, then $\angle ACD > \angle ACB = 90^\circ$, and ACD is not a right triangle. If, on the other hand, C and D lie on opposite sides of AB, then $\angle DBC > \angle DBA = 90^\circ$, so DBC is not a right triangle. Thus, either all points lie on the circle, or they all lie on the two perpendiculars. In either case, there can be at most 4 such points.

Solution 2. b) Assume by contradiction that it is possible to choose 5 points satisfying the conditions. Since each three points form the vertices of a right triangle, there are 10 right triangles with vertices in these 5 points. Thus, there exists a point O that is the vertex of at least two right angles. Let OAB and OXY be the two triangles with right angles at O.

Now, if either X or Y was lying on line OA, the other point would have to lie on AB. But then we would have three points on the same line, since at most one of X and Y can coincide with A or B. Analogously, neither X nor Y can lie on OB. Now if X (resp. Y) and B lie on opposite sides of line OA, then XOB (resp. YOB) is an obtuse triangle. Similarly, X and A (or Y and A) cannot lie on opposite sides of OB. Thus, both X and Y must lie within the right angle AOB, but then XOY is not a right triangle.

FR-7. Call a rectangle *splittable* if it can be divided into two or more square parts such that the side of each square is of integral length and there is a unique square with smallest side length. Find the dimensions of the splittable rectangle with the least possible area. *(Grade 11.)*

Answer: 5×7.

Solution. The unique smallest square of the partition cannot lie on the side of the rectangle, for it would have a larger square on either side and the area between the two squares could only be filled by squares no larger than the smallest square. Analogously, the smallest square cannot lie in the corner. Now, the distance between the smallest square and any side of the triangle must be at least one unit longer than the side length of the smallest square, for otherwise the area between the smallest square and the side could not be filled. Thus, the length of each side of the rectangle is at least $1 + 2 + 2 = 5$ and each square on a side must have side length at least 2. Thus, if the rectangle has a side of length 5, on this side we must have a square with side length at least 3. But then the distance between the smallest square and this side is at least 3. The same holds for the opposite side of length 5, so the length of the longer side must be at least $1 + 3 + 3 = 7$. It is possible to partition a 5×7 rectangle in the desired way (see Fig. 13). The area of this rectangle is 35, which is indeed the smallest possible area, since any rectangle with shorter side length greater than 5 has area at least $6 \cdot 6 = 36$.

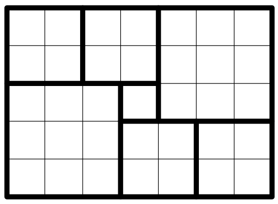


Figure 13

FR-8. Circles c_1 and c_2 with respective diameters AB and CD of different length touch externally at point K. An external tangent common to both circles touches c_1 at A and c_2 at C. Line BD intersects c_1 again at point L and c_2 at point M. Prove that triangles

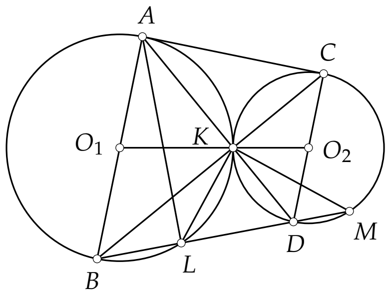


Figure 14

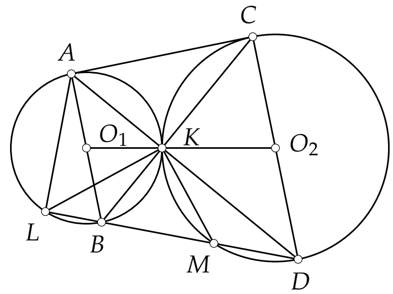


Figure 15

AKL and BKM are similar. *(Grade 11.)*

Solution. Let O_1 and O_2 be the midpoints of circles c_1 and c_2, respectively (Fig. 14 and 15). The isosceles triangles BO_1K and CO_2K are similar, since their corresponding legs are parallel: $AB \parallel CD$ and point K lies on O_1O_2. Thus, the bases are also parallel, so K lies on BC. Now on one hand, $\angle KAL = \angle KBM$, while on the other hand $\angle ALK = \angle ABK = \angle KCD = \angle KMB$. We see that triangles AKL and BKM have two pairs of equal angles and hence are indeed similar.

FR-9. Let a, b, c be real numbers. Prove that $a^2 + 4b^2 + 8c^2 \geqslant 3ab + 4bc + 2ca$. When does equality hold? *(Grade 11.)*

Answer: Equality holds iff $a = 2b = 4c$.
Solution. Bringing all terms to the lhs, we get

$$\begin{aligned} &a^2 + 4b^2 + 8c^2 - 3ab - 4bc - 2ca = \\ &\quad = \left(\frac{3}{4}a^2 - 3ab + 3b^2\right) + (b^2 - 4bc + 4c^2) + \left(4c^2 - 2ca + \frac{1}{4}a^2\right) = \\ &\quad = \left(\frac{\sqrt{3}}{2}a - \sqrt{3}b\right)^2 + (b - 2c)^2 + \left(2c - \frac{1}{2}a\right)^2 \geqslant 0. \end{aligned}$$

Equality holds iff equations $\frac{\sqrt{3}}{2}a = \sqrt{3}b$, $b = 2c$, $2c = \frac{1}{2}a$ hold simultaneously, in other words, iff $a = 2b = 4c$.
Remark. One may find other solutions, precisely using AM-GM on $(1.5a^2, 6b^2)$, $(0.5a^2, 8c^2)$, $(2b^2, 8c^2)$, grouping the lhs as $\left(a - \frac{3}{2}b - c\right)^2 + \left(\frac{\sqrt{7}}{2}b - \sqrt{7}c\right)^2$, or considering the lhs as a quadratic trinomial in a, b and c and investigating the respective discriminants.

FR-10. Does there exist a convex hexagon $ABCDEF$ such that the circumcircles of triangles ABC, CDE and EFA intersect at a common point inside the hexagon? *(Grade 11.)*

Answer: no.

Solution. Suppose that such a hexagon exists and let O be the common intersection point of the three circumcircles (Fig. 16). Then quadrilaterals $ABCO$, $CDEO$ and $EFAO$ are all inscribed, so $\angle BAO + \angle BCO = 180^\circ$, $\angle DCO + \angle DEO = 180^\circ$ and $\angle FEO + \angle FAO = 180^\circ$. Adding the three equations, we get $\angle BCD + \angle DEF + \angle FAB = 3 \cdot 180^\circ$. On the other hand, all internal angles of a convex hexagon are less than 180°, so the sum of the three angles cannot reach $3 \cdot 180^\circ$, contradiction.

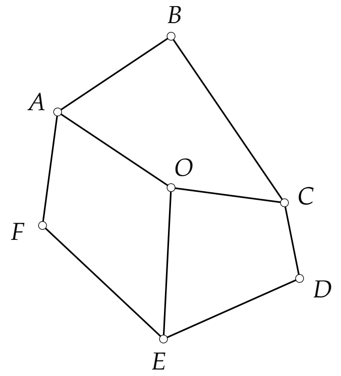


Figure 16

FR-11. Find the least possible value of $(1+u^2)(1+v^2)$, where u and v are real numbers satisfying $u+v=1$. *(Grade 12.)*

Answer: $\frac{25}{16}$.

Solution 1. Write $u = \frac{1}{2} + x$ and $v = \frac{1}{2} - x$. Then

$$(1+u^2)(1+v^2) = \left(1 + \left(\frac{1}{2}+x\right)^2\right)\left(1 + \left(\frac{1}{2}-x\right)^2\right) = \left(1 + \frac{1}{4} + x^2 + x\right) \cdot$$
$$\cdot \left(1 + \frac{1}{4} + x^2 - x\right) = \left(\frac{5}{4} + x^2\right)^2 - x^2 = \frac{25}{16} + \frac{5}{2}x^2 + x^4 - x^2 = \frac{25}{16} + \frac{3}{2}x^2 + x^4.$$

Since $\frac{3}{2}x^2$ and x^4 are both non-negative, the obtained sum is minimal when $x = 0$. The latter gives $(1+u^2)(1+v^2) = \frac{25}{16}$.

Solution 2. As $u+v=1$, we get

$$\begin{aligned}(1+u^2)(1+v^2) &= 1 + u^2 + v^2 + u^2v^2 = \\ &= 1 + (u+v)^2 - 2uv + u^2v^2 = 2 - 2(uv) + (uv)^2.\end{aligned}$$

Let $s = uv$. For a fixed sum $u+v=1$, the product $s = uv$ is maximal when $u = v$. Thus, we can bound $s \leqslant \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Now, we need to minimize $2 - 2s + s^2 = (s-1)^2 + 1$, which is decreasing in $\left(-\infty; \frac{1}{4}\right]$ and obtains the minimum at $s = \frac{1}{4}$.

Solution 3. Notice that $u = 1$, $v = 0$ gives $(1+u^2)(1+v^2) = 2$, while for any $u > 1$ or $v > 1$ (or equivalently, $v < 0$ or $u < 0$), $(1+u^2)(1+v^2) > 2$. Thus, we may restrict to the case $u, v \in [0,1]$. Now consider a triangle ABC such that its side BC and altitude AH (Fig. 17) have unit length and H divides BC to parts of length u and v. Then $u+v=1$ and the law of sines gives $\frac{1}{2} \cdot |AB| \cdot |AC| \cdot \sin \angle BAC = \frac{1}{2} \cdot |BC| \cdot |AH| = \frac{1}{2}$, so $(1+u^2)(1+v^2) = |AB|^2|AC|^2 = \frac{1}{\sin^2 \angle BAC}$. The value $\sin \angle BAC$ is maximal when H is the midpoint of BC. Indeed, let c be the circumcircle of ABC in the case when H is the midpoint. For any other configuration, A lies outside this circle c and thus the angle BAC is smaller (note that the angle BAC is always acute as BC cannot be the longest side of ABC). Now if H is the midpoint of BC, we get $|AB|^2 = |BC|^2 = \frac{5}{4}$, and

$|AB|^2|BC|^2 = \dfrac{25}{16}$.

Remark. One may find other solutions, precisely determining the minima of $g(u) = (1+u^2)(1+(1-u)^2)$ using derivatives, or writing out Jensen's inequality for a convex function $l(x) = \ln(1+x^2)$.

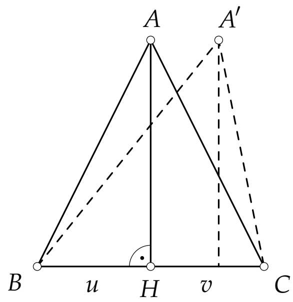


Figure 17

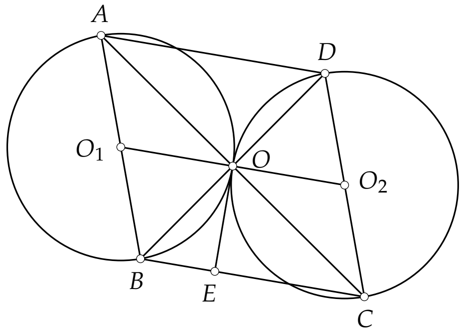


Figure 18

FR-12. In a convex quadrilateral $ABCD$, $|AB| = |BC| = |CD|$. Diagonals AC and BD intersect at point O. Prove that the circumcircles of triangles AOB and COD are mutually tangent iff AC is perpendicular to BD. *(Grade 12.)*

Solution. Assume first that the circumcircles of AOB and COD are mutually tangent (Fig. 18). Draw a tangent common to both circles from O, and let the tangent line intersect BC at E. Then $|AB| = |BC|$ implies $\angle EOB = \angle OAB = \angle BCO$. Similarly, $\angle EOC = \angle ODC = \angle CBO$. Triangle BOC now gives $\angle EOB + \angle EOC + \angle OBC + \angle OCB = 180^\circ$ or $2\angle EOB + 2\angle EOC = 180^\circ$, so finally $\angle BOC = \angle EOB + \angle EOC = 90^\circ$.

Assume now AC is perpendicular to BD. The circumcentres O_1 and O_2 of right triangles AOB and COD lie on the respective hypotenuses AB and CD. We have $\angle O_1OA = \angle O_1AO = \angle BCO$ and $\angle O_2OD = \angle O_2DO = \angle CBO$. As BOC is also a right triangle, $\angle BCO + \angle CBO = 90^\circ$. Finally, $\angle O_1OA + \angle AOD + \angle O_2OD = \angle BCO + 90^\circ + \angle COB = 180^\circ$, so the circumcircles of AOB and COD touch at O.

FR-13. All natural numbers that are less than a fixed positive integer n and relatively prime to it are added one-by-one in increasing order. How many intermediate sums (starting from the lonely first addend and including the final sum) are divisible by n, if

a) n is an odd prime number?

b) n is the square of an odd prime number?

(Grade 12.)

Answer: a) 1; b) 1.

Solution. a) Let $n = p$ where p is an odd prime. The addends are $1, 2, \ldots, p-1$, thus the intermediate sums have form $1 + \ldots + k$ where $1 \leqslant k \leqslant p-1$. Suppose $p \mid 1 + \ldots + k$. Then $p \mid k(k+1)$ as $1 + \ldots + k = \dfrac{k(k+1)}{2}$. Thus either $p \mid k$ or $p \mid k+1$. This is possible only for $k = p-1$.

b) Let $n = p^2$ where p is an odd prime. Any number less than p^2 is added if and only if it is not divisible by p. Divide the addends into p groups, each consisting of $p-1$

members:

$$\begin{array}{cccc} 1, & 2, & \ldots\ldots, & p-1, \\ p+1, & p+2, & \ldots\ldots, & p+p-1, \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ (p-1)p+1, & (p-1)p+2, & \ldots\ldots, & (p-1)p+p-1. \end{array}$$

Let an intermediate sum be divisible by p^2; then it is divisible by p, too. As the rows are equivalent modulo p, we can use part a) of the problem to deduce that the last intermediate sum of every row is divisible by p and the others are not. Hence, in the whole intermediate sum under consideration, the last row cannot occur partially, i.e., our intermediate sum consists of whole rows of addends.

The sum of the elements of the first row is $\dfrac{p(p-1)}{2}$. The sum of the numbers of each following row is by $p(p-1)$ larger than that of the row preceding it. Thus the row sums are $1 \cdot \dfrac{p(p-1)}{2}, 3 \cdot \dfrac{p(p-1)}{2}, 5 \cdot \dfrac{p(p-1)}{2}$ etc.. The sum of the numbers of the first i rows is $(1+3+\ldots+(2i-1)) \cdot \dfrac{p(p-1)}{2} = i^2 \cdot \dfrac{p(p-1)}{2}$.

If $p^2 \mid i^2 \cdot \dfrac{p(p-1)}{2}$ then $p \mid i^2 \cdot (p-1)$, implying $p \mid i$. Hence $i = p$, i.e., the sum is the final sum.

Remark. It is easy to show that the entire sum of $\varphi(n)$ addends is divisible by n for all integers $n > 2$. If n is neither a prime nor the square of a prime then there can be more intermediate sums divisible by n. For example, if $n = 16$ then the intermediate sum $1+3+5+7$ containing only half of the addends is divisible by 16. If $n = 27$ or $n = 39$ then two intermediate sums in addition to the final sum are divisible by n, etc.

FR-14. Consider a point X on line l and a point A outside the line. Prove that if there exists a point Z_1 on l such that the three side lengths of triangle AXZ_1 are all rational, then there exist two other points Z_2 and Z_3 on l such that the side lengths of triangles AXZ_2 and AXZ_3 are also all rational. *(Grade 12.)*

Solution. We consider three separate cases.

- If AXZ_1 is equilateral, i.e., $|AX| = |AZ_1|$ and $\angle XAZ_1 = 60^\circ$, then take Z_2 on the extension of Z_1X across X such that $|XZ_2| = 0.6\,|AX|$, and take Z_3 to be the reflection of Z_2 across the perpendicular bisector of XZ_1 (Fig. 19). The law of cosines implies $|AZ_2| = |AX|^2 + |XZ_2|^2 - 2 \cdot |AX| \cdot |XZ_2| \cdot \cos 120^\circ = 1.96\,|AX|^2$, so $|AZ_2| = 1.4\,|AX|$ and the side lengths of AXZ_2 as well as AXZ_3 are rational.

Let now AXZ_1 be not equilateral.

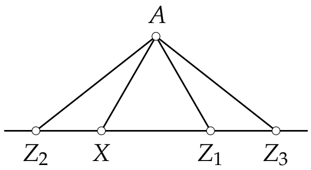


Figure 19

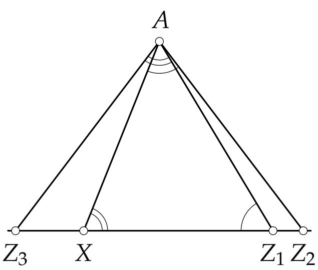


Figure 20

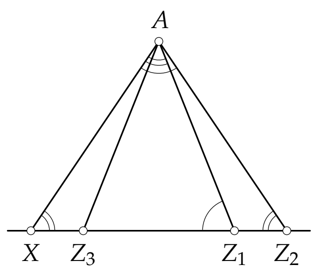


Figure 21

- Assume $|XZ_1| \neq |AX|$ and $|XZ_1| \neq |AZ_1|$. Choose Z_2 on ray XZ_1 such that $\angle Z_2AX = \angle AZ_1X$, and choose Z_3 on ray Z_1X such that $\angle Z_3AZ_1 = \angle AXZ_1$ (Fig. 20). Points Z_2 and Z_1 differ since $\angle AZ_1X \neq \angle Z_1AX$, and points Z_3 and X differ since $\angle AXZ_1 \neq \angle XAZ_1$. Triangles Z_2XA and Z_3AZ_1 are similar to triangle AXZ_1 with similarity ratios $\frac{|AX|}{|Z_1X|}$ and $\frac{|Z_1A|}{|Z_1X|}$, so their side lengths are rational, and $|XZ_3|$ is rational, too.
- Assume w.l.o.g. $|XZ_1| = |AZ_1|$ (Fig. 21). Choose Z_2 as before, then Z_2 differs from X and Z_1 and the side lengths of AXZ_2 are rational. Take Z_3 to be the reflection of Z_1 across the perpendicular bisector of XZ_2. Then Z_3 differs from Z_1, as AXZ_1 is not an isosceles right triangle. Triangle AXZ_3 is equal to triangle AZ_2Z_1, and the latter has rational side lengths.

FR-15. A finite number of thin straight pins are attached to a vertical wall such that no two pins touch each other. If a pin is detached, it slides straight down the wall, keeping its original angle to the floor. Prove that there exists a pin that can slide freely down to the floor without being stopped by any of the other pins. *(Grade 12.)*

Solution 1. If there exists a vertical pin that can slide freely, we are done. Assume now that no vertical pin can slide down freely. Draw a horizontal line l where the wall meets the floor and project the endpoints of each pin onto l. If there are no pin points between some left endpoint and l, colour the projection point on l blue. Similarly, if some right endpoint is the lowermost pin point on its projection line, colour the corresponding projection point yellow. Clearly, the leftmost projection point on l is coloured blue, while the rightmost point is yellow. Thus, moving on l left-to-right, some two consecutive coloured points must be blue and yellow, respectively. We claim that these points are the two endpoints of the same pin, and thus this pin can slide down. Indeed, on the segment between the blue and the yellow point, any lowest pin point above l must belong to the same pin as the left (blue) and the right (yellow) endpoint.

Solution 2. We prove by induction on the number of pins. The claim clearly holds for one pin. Assume there is more than one pin, and consider three cases.

1. There exists a pin p such that below every point of p, there is a point of some other pin. Remove p, then by the induction assumption, some pin v can slide down freely. Now, put p back. Then p cannot be the only pin stopping v, since below every point of p, there is a point of another pin, and at least one of those pin points should also be stopping v.
2. There exists a pin p which cannot slide down freely such that all pins stopping p lie entirely below p. Remove p and all the remaining pins that do not lie below p. Then, there must exist a pin v that can slide down freely, but then v can also slide down in the original configuration, since the only pins possibly stopping it are those below p.
3. If the previous two cases do not hold, then each pin has some points that have no other pin points below them, and either the pin can slide down or one of the pins stopping it does not lie entirely below this pin. Let p be the pin with the rightmost point amongst all pins (if there is more than one such pin, choose the one with the topmost such point). Remove p. By the induction assumption, there now exists a

pin v that can slide down. Put p back. If v can still slide down, we are done. In the other case, the only pin stopping it is p. Since v must have some free points with no other pin points below, the left endpoint of v must reach further left than the left endpoint of p. We claim that now p can slide down. Indeed, any pin points below p that also lie below v cannot be stopping p, as they would also be stopping v. But any other pin below p can also not be stopping p, as p has the rightmost endpoint, so any pins stopping p should lie completely below p.

Remark 1. One can find other solutions, precisely using a directed graph with pins as its vertices and an edge from vertex a to vertex b if pin b is stopping pin a: it suffices to prove that this graph does not contain any directed cycles.
Remark 2. The claim does not always hold when the pins are not straight. For example, two half-circle pins can be placed to mutually stop each other.

IMO team selection contest

First day

TS-1. There are 2008 participants in a programming competition. In every round, all programmers are divided into two equal-sized teams. Find the minimal number of rounds after which there can be a situation in which every two programmers have been in different teams at least once.

Answer: 11.
Solution 1. After every round consider the biggest set of programmers where the programmers have been in the same team in all rounds so far. Before the first round it consists of 2008 programmers. With every round its size can decrease by at most twice, since the programmers belonging to it are divided among two teams in the new round and at least half of them will again be in the same team. Thus the number of rounds is at least $\log_2 2008$, i.e. at least 11.
We shall show that 11 rounds suffice. Order the 2008 programmers in some way and add both at the end and at the beginning 20 imaginary programmers. Number the programmers by 11-digit binary numbers from 0 to 2047, adding leading zeros if necessary. In round i the programmers are divided into teams according to the ith digit of their number. In every round the kth imaginary programmer from the beginning and the kth imaginary programmer from the end are in different teams since their corresponding binary numbers have all digits different. Hence both teams have in every round an equal number of programmers. Also, every pair of programmers belong to different teams in at least one round since their numbers differ in at least one binary digit.
Solution 2. Let us prove by induction on k that if the number of programmers $2n$ satisfies the inequalities $2^{k-1} < 2n \leqslant 2^k$ then k rounds suffice. If $k = 1$ then we have 2 programmers and clearly one round is enough. Assume the claim is true for some k. Assume there are $2n$ programmers where $2^k < 2n \leqslant 2^{k+1}$. Divide them into two groups of $s = 2^k$ and $t = 2n - s$ programmers, and number them by $1, \ldots, s$ and $s + 1, \ldots, s + t$ respectively. By the induction hypothesis the programmers in both the first and the second group can be divided into equal-sized teams so that after k rounds every two (in

each group) have competed against each other at least once. For rounds $1, \dots, k$ make up two new equal teams by taking one team corresponding to each group and putting them together. Assume without loss of generality that in round k one team consists of the programmers of the first group with numbers $1, \dots, \frac{s}{2}$ and of the second group with numbers $s+1, \dots, s+\frac{t}{2}$. In the new round swap and make up a team of programmers with numbers $1, \dots, \frac{s}{2}$ and $s+\frac{t}{2}+1, \dots, s+t$. We can check that every two programmers (also from different groups) have now been in different teams at least once.
The other part can be done like in Solution 1.

TS-2. Let $ABCD$ be a cyclic quadrangle whose midpoints of diagonals AC and BD are F and G, respectively.

a) Prove the following implication: if the bisectors of angles at B and D of the quadrangle intersect at diagonal AC then $\frac{1}{4} \cdot |AC| \cdot |BD| = \sqrt{|AG| \cdot |BF| \cdot |CG| \cdot |DF|}$.

b) Does the converse implication also always hold?

Answer: b) No.
Solution 1. a) Let E be the intersection point of the bisectors from B and D. By the bisector property,

$$\frac{|AB|}{|BC|} = \frac{|AE|}{|EC|} = \frac{|AD|}{|DC|}. \tag{1}$$

By Ptolemy's theorem, $|AB| \cdot |CD| + |AD| \cdot |BC| = |AC| \cdot |BD|$. Using this in (1), we obtain

$$2 \cdot |BC| \cdot |AD| = |AC| \cdot |BD|, \tag{2}$$

$$2 \cdot |AB| \cdot |CD| = |AC| \cdot |BD|. \tag{3}$$

Let F be the midpoint of AC. Then $\angle FAD = \angle CAD = \angle CBD$. By (2), $\frac{|FA|}{|AD|} = \frac{|AC|}{2|AD|} = \frac{|BC|}{|BD|}$. Hence triangles FAD and CBD are similar. Analogously by (3), triangles FAB and CDB are similar. Consequently, triangles FAD and FBA are similar. Thus $\frac{|FA|}{|FD|} = \frac{|FB|}{|FA|}$ which implies

$$\frac{1}{4}|AC|^2 = |FB| \cdot |FD|. \tag{4}$$

By (1), $\frac{|DA|}{|AB|} = \frac{|DC|}{|CB|}$. Thus bisector property implies that the bisectors of angles at A and C intersect at diagonal BD. Let G be the midpoint of BD. Analogously to what we did before, we obtain

$$\frac{1}{4}|BD|^2 = |AG| \cdot |CG|. \tag{5}$$

The desired claim follows now by multiplying the corresponding sides of (4) and (5) and finding the square root.
b) Let $ABCD$ be a rectangle where $|AB| > |BC|$. Clearly $|AG| = |BF| = |CG| = |DF| = \frac{1}{2}|AC| = \frac{1}{2}|BD|$, implying the rhs of the implication of part a) (lhs of the converse). But $\frac{|AB|}{|BC|} > 1 > \frac{|AD|}{|DC|}$ shows that the bisectors of angles at B and D do not intersect on diagonal AC. Hence the converse implication is false.
Solution 2. a) Denote the interior angles of $ABCD$ by $\angle A$, $\angle B$, $\angle C$, $\angle D$. In triangle DAB, cosine law gives

$$|BD|^2 = |AB|^2 + |AD|^2 - 2 \cdot |AB| \cdot |AD| \cdot \cos\angle A.$$

In triangle BCD, taking into account that $\angle C = 180^\circ - \angle A$, cosine law gives

$$|BD|^2 = |CB|^2 + |CD|^2 + 2 \cdot |CB| \cdot |CD| \cdot \cos\angle A.$$

Multiplying these two equalities leads to

$$|BD|^4 = (|AB|^2 + |AD|^2)(|CB|^2 + |CD|^2) - 4 \cdot |AB| \cdot |AD| \cdot |CB| \cdot |CD| \cdot \cos^2\angle A + \\ + 2\left((|AB|^2 + |AD|^2) \cdot |CB| \cdot |CD| - (|CB|^2 + |CD|^2) \cdot |AB| \cdot |AD|\right) \cdot \cos\angle A.$$

On the other hand, $2\overrightarrow{AG} = (\overrightarrow{AB} + \overrightarrow{AD})$ implies

$$4 \cdot |AG|^2 = |AB|^2 + |AD|^2 + 2 \cdot |AB| \cdot |AD| \cdot \cos\angle A$$

and, analogously (using $\angle C = 180^\circ - \angle A$),

$$4 \cdot |CG|^2 = |CB|^2 + |CD|^2 - 2 \cdot |CB| \cdot |CD| \cdot \cos\angle A.$$

Multiplying these equalities leads to

$$16 \cdot |AG|^2 \cdot |CG|^2 = \\ = (|AB|^2 + |AD|^2)(|CB|^2 + |CD|^2) - 4 \cdot |AB| \cdot |AD| \cdot |CB| \cdot |CD| \cdot \cos^2\angle A - \\ - 2\left((|AB|^2 + |AD|^2) \cdot |CB| \cdot |CD| - (|CB|^2 + |CD|^2) \cdot |AB| \cdot |AD|\right) \cdot \cos\angle A.$$

If the bisectors of angles by B and D intersect on diagonal AC, the bisector property gives $\frac{|AB|}{|CB|} = \frac{|AD|}{|CD|}$ or $|AB| \cdot |CD| = |AD| \cdot |CB|$. Thus

$$(|AB|^2 + |AD|^2) \cdot |CB| \cdot |CD| - (|CB|^2 + |CD|^2) \cdot |AB| \cdot |AD| = \\ = |AB| \cdot |AD| \cdot |CB|^2 + |AD| \cdot |AB| \cdot |CD|^2 - (|CB|^2 + |CD|^2) \cdot |AB| \cdot |AD| = 0.$$

Consequently, $|BD|^4 = 16 \cdot |AG|^2 \cdot |CG|^2$. Considering triangles ABC and CDA, we obtain in a similar way that $|AC|^4 = 16 \cdot |BF|^2 \cdot |DF|^2$. Multiplying the last equalities and taking the 4th root from both, we obtain the desired result.

TS-3. Let n be a positive integer and x, y positive real numbers such that $x^n + y^n = 1$. Prove the inequality

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

Solution. Note first that $\dfrac{1+x^{2k}}{1+x^{4k}} < \dfrac{1}{x^k}$. Indeed,

$$\frac{1+x^{2k}}{1+x^{4k}} - \frac{1}{x^k} = \frac{x^k + x^{3k} - 1 - x^{4k}}{(1+x^{4k})x^k} = \frac{(x^{3k}-1)(1-x)}{(1+x^{4k})x^k} < 0,$$

since the conditions of the problem imply $0 < x < 1$. Now we estimate

$$\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}} < \sum_{k=1}^{n} \frac{1}{x^k} = \frac{x^n - 1}{x^n(x-1)}.$$

A similar inequality can be proven for y, so we obtain

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \left(\sum_{k=1}^{n} \frac{1}{x^k}\right)\left(\sum_{k=1}^{n} \frac{1}{y^k}\right) =$$

$$= \frac{x^n - 1}{x^n(x-1)} \cdot \frac{y^n - 1}{y^n(y-1)} = \frac{1}{(x-1)(y-1)}.$$

Remark. This problem, proposed by Estonia, appeared in the IMO-2007 Shortlist.

Second day

TS-4. Sequence (G_n) is defined by $G_0 = 0$, $G_1 = 1$ and $G_n = G_{n-1} + G_{n-2} + 1$ for every $n \geqslant 2$. Prove that for every positive integer m there exist two consecutive terms in the sequence that are both divisible by m.

Solution. Define $G_{-1} = 0$, then $G_n = G_{n-1} + G_{n-2} + 1$ holds also when $n = 1$. Consider the pairs (G_n, G_{n+1}) of consecutive members of the sequence. There are only m^2 pairs modulo m, hence there are pairs (G_k, G_{k+1}) and (G_l, G_{l+1}) with $k < l$ that are componentwise congruent modulo m. Since $G_{n-2} = G_n - G_{n-1} - 1$, two consecutive terms in the sequence determine the previous term uniquely. The same is true modulo m. Therefore also pairs (G_{k-1}, G_k) and (G_{l-1}, G_l) are componentwise congruent modulo m. Continuing, we see that (G_{-1}, G_0) and (G_{l-k-1}, G_{l-k}) are componentwise congruent modulo m. Since $G_{-1} = G_0 = 0$, the terms G_{l-k-1} and G_{l-k} are divisible by m as required.

TS-5. Points A and B are fixed on a circle c_1. Circle c_2, whose centre lies on c_1, touches line AB at B. Another line through A intersects c_2 at points D and E, where D lies between A and E. Line BD intersects c_1 again at F. Prove that line EB is tangent to c_1 if and only if D is the midpoint of the segment BF.

Solution 1. Let K be the second intersection point of the line AD and the circle c_1 (Fig. 22). The triangles KFD and BAD are similar since the corresponding angles are equal. The triangle BAD is similar to the triangle EAB since, by tangent-secant theorem, $\angle ABD = \angle BED$ and they have a common angle at the vertex A. Let O be the centre of the circle c_2. Since AB is the tangent to the circle c_2 at point B, $AB \perp BO$. It follows that AO is a diameter of the circle c_1 since O is on the circle c_1 by assumption. Hence also $OK \perp AK$ from which it follows that OK is an altitude of the isosceles triangle ODE. Thus $|DK| = |KE|$.

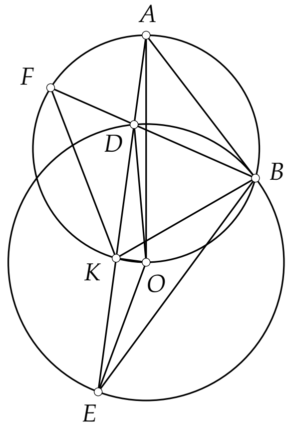


Figure 22

The line EB is tangent to the circle c_1 at B if and only if $\angle EBK = \angle BAD$. Since $\angle ABD = \angle BED$, the last equality is equivalent to the triangles EKB and BAD being similar. By the same equality of angles, the two triangles are similar if and only if $\frac{|AB|}{|BE|} = \frac{|DB|}{|KE|}$. Since EAB and KFD are similar triangles, the last equality is equivalent to $\frac{|FD|}{|DK|} = \frac{|DB|}{|KE|}$. This is equivalent to $|FD| = |DB|$ since the denominators are equal.

Solution 2. As in the first solution we show that $|DK| = |KE|$. Let $|AD| = x$, $|DK| = |KE| = y$, $|BE| = z$, $|DB| = u$, $|FD| = v$, $|AB| = w$. By the property of intersecting chords, $uv = xy$. Since AB is a tangent, $w^2 = x(x+2y)$. The triangles ABD and AEB are similar since $\angle ABD = \angle BED$ and at vertex A they have a common angle. Thus $\frac{u}{x} = \frac{z}{w}$ and hence $z = \frac{uw}{x}$.

The condition that the line EB is tangent to the circle c_1 is equivalent to $z^2 = y(x+2y)$. We shall show that the last condition is equivalent to $u = v$:

$$z^2 = y(x+2y) \;\Leftrightarrow\; \frac{u^2w^2}{x^2} = y(x+2y) \;\Leftrightarrow\; u^2x(x+2y) = x^2y(x+2y) \;\Leftrightarrow$$
$$\Leftrightarrow\; u^2 = xy \;\Leftrightarrow\; u^2 = uv \;\Leftrightarrow\; u = v.$$

TS-6. A *string of parentheses* is any word that can be composed by the following rules.

1) () is a string of parentheses.
2) If s is a string of parentheses then (s) is a string of parentheses.
3) If s and t are strings of parentheses then st is a string of parentheses.

The *midcode* of a string of parentheses is the tuple of natural numbers obtained by finding, for all pairs of opening and its corresponding closing parenthesis, the number of characters remaining to the left from the medium position between these parentheses, and writing all these numbers in non-decreasing order. For example, the midcode of $(())$ is $(2,2)$ and the midcode of $()()$ is $(1,3)$. Prove that midcodes of arbitrary two different strings of parentheses are different.

Solution. We can assume that the two strings have equal lengths because otherwise their midcodes differ by length. We prove the desired claim by induction on the length. In the case of length 2, the claim holds trivially. Let s and t be two longer strings of parentheses. Consider, for both of them, the longest prefix that forms a string of parentheses itself. The first and the last character of such prefix form a pair of opening and corresponding closing parenthesis.

If the prefixes of s and t under consideration have different lengths $2k$ and $2l$, respectively, where assume w.l.o.g. that $k < l$, then consider the first k numbers in the midcodes of both strings. Let the opening parentheses occur at positions $a_1, \ldots, a_k$ and the corresponding closing parentheses occur at positions $b_1, \ldots, b_k$ in word s. The number in midcode that corresponds to the ith pair of parentheses is $\frac{a_i + b_i - 1}{2}$. As the first $2k$ characters of s form a string of parentheses, numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ are precisely $1, \ldots, 2k$ in some order. Thus the sum of k smallest members of the midcode of s is

$$\sum_{i=1}^{k} \frac{a_i + b_i - 1}{2} = \frac{1 + 2 + \ldots + 2k}{2} - \frac{k}{2}.$$

In the midcode of t, the sum of k smallest members is larger since, otherwise, the sum of position indices of some k pairs of parentheses would be $1 + 2 + \ldots + 2k$. This would imply that the corresponding closing parenthesis for each opening parenthesis among those at positions $1, 2, \ldots, 2k$ occurs within the same positions, leading to $l \leqslant k$, a contradiction.

If both prefixes under consideration have length $2k$ then, for both cases, the part of the word between the first and the last character of the prefix forms a string of parentheses, as does the part of the word remaining after the prefix (provided they are non-empty). As s and t differ, either the first mentioned parts of the words or the second mentioned parts differ.

In the former case, the induction hypotheses implies that their midcodes also differ. In the midcodes of s and t, these midcodes are represented by numbers that are by 1 larger, whereby all these numbers are less than $2k$. In addition, both midcodes contain k (from the pair of parentheses embracing the prefix) and the remaining numbers are larger than $2k$. Thus the midcodes of s and t differ.

In the latter case, the induction hypothesis again implies that the midcodes of the parts after the prefix are different. In the midcodes of s and t, these midcodes are represented by numbers that are by $2k$ larger. All other numbers in the midcode are less than $2k$. Hence the midcodes differ.

Remark. A tuple of positive integers $x_1, \ldots, x_n$ is a midcode of some string of parentheses iff it is monotone, $\sum_{i=1}^{k} x_i \geqslant k^2$ for every $k = 1, \ldots, n$, and $\sum_{i=1}^{n} x_i = n^2$.

Problems listed by topics

Number theory: OC-2, OC-3, OC-4, OC-9, OC-14, FR-13, TS-4
Algebra: OC-6, OC-11, OC-13, FR-1, FR-2, FR-9, FR-11, TS-3
Geometry: OC-7, OC-12, FR-3, FR-5, FR-8, FR-10, FR-12, FR-14, TS-2, TS-5
Discrete mathematics: OC-1, OC-5, OC-8, OC-10, FR-4, FR-6, FR-7, FR-15, TS-1, TS-6